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Government Finance in a Model of Currency Substitution

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Abstract

Our model is a variant of the cash-in-advance model. Goods must be purchased in the seller's currency, but currency may be traded before shopping at a cost. This cost is a measure of substitutability. The model is applied to seignorage taxation. We show that optimal money growth is positive and increasing in substitutability if and only if first- and second-period consumption are gross substitutes. If governments act independently, money growth is suboptimally low if currencies are sufficiently substitutable and too high otherwise.

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Summary

This paper develops an equilibrium, optimizing model of currency substitution and applies it to seigniorage taxation in the presence of currency substitution. The theoretical framework is an overlapping-generations model with the finance constraint that goods must be purchased in the seller's currency. When consumers are young they are uncertain about what their preferences will be when they are old. Thus, they must decide how much of each currency to hold without knowing their future demand for goods. Consumers may trade currencies at a cost in the spot market immediately before purchasing goods. This cost is a measure of the substitutability of currencies. When the transaction cost is infinite, the model becomes a cash-in-advance constraint model; when the transaction cost is zero, the currencies are perfect substitutes and the analysis replicates the indeterminate-exchange-rate model of Kareken and Wallace (1981).

The model is found to yield money demand functions with reasonable properties. Demand for both currencies is positive when the cost of transacting in the spot market exceeds the rate of depreciation of the weaker currency. An increase in the future price of country two's good relative to country one's good lowers the share of portfolios allocated to the currency of country one if and only if first- and second-period consumption are gross substitutes. If utility is log-linear, the share of portfolios allocated to the weaker (stronger) currency depends positively (negatively) upon the transaction cost. The share of portfolios allocated to either currency depends positively upon its rate of appreciation. A stationary equilibrium is shown to exist if the cost of spot market transactions is not too small and monetary policies are not too different.

Governments are assumed to finance their expenditures by levying a costly income tax or by collecting seigniorage. In deciding the optimal level of inflation, they weigh the trade-offs between the costs of administering and complying with income taxes and the distortions created by rising prices. If identical governments cooperate, then the optimal rate of money growth is strictly positive. It is decreasing in the substitutability of currencies if and only if first- and second-period consumption are gross substitutes. If governments act independently, money growth may be either too high or too low depending on the degree of currency substitution.

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I. Introduction

The liberalization of capital flows and financial integration in the European Community and elsewhere has prompted new interest in currency substitution. The term "currency substitution" has been used in a variety of ways in the literature. Here it is taken to mean the use of multiple currencies within a single country as a store of value. 1/

This view is similar to that of Handa (1988) and Elkhafif and Kubursi (1991). It is also consistent with Tanzi and Blejer (1982) who consider it to be where domestic money demand depends on foreign variables and with Ramirez-Rojas (1986) who regards it as domestic demand for foreign currency. 2/ This paper develops an equilibrium, optimizing model of currency substitution and applies it to the issue of seignorage taxation.

Our theoretical framework is a variant of the cash-in-advance model. Goods must be purchased with the currency of the seller's home country, but we assume that currencies may be traded - at a cost - in the spot market immediately prior to purchasing the consumption goods. This cost is a measure of the substitutability of currencies. When the transactions cost is infinite, the model becomes a standard cash-in-advance constraint model; when the transactions cost is zero the currencies are perfect substitutes and our analysis replicates the indeterminate-exchange-rate model of Kareken and Wallace (1981).

To keep matters simple, the model has no aggregate uncertainty, although allowing for random outputs and money growth rates would be a simple extension. However, in order to ensure the possibility that both currencies are valued, consumers may have a precautionary demand for the weaker currency because of uncertainty about their future preferences.

Perhaps the most common way to model currency substitution is to specify money demand functions or to put money in the utility function. 3/ These approaches are not used here because they do not capture any real resource costs associated with the lack of substitutability of currencies. Moreover, because the measure of substitutability is a parameter of the

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1/ Giovannini and Turtelboom (forthcoming) survey numerous models of currency substitution.

2/ Calvo and Végh (1992) define the concept more strictly as the use of multiple currencies as media of exchange within one country. In a developing country, they refer to the use of a foreign currency as a store of value as "dollarization".

3/ Girton and Roper (1981) impose money demand functions which depend upon returns to alternative monies. Weil (1981) and Sturzenegger (1992) put home and foreign money directly in the utility function. Woodford (1991) notes that cash-in-advance constraints make indirect utility a function of home and foreign real balances. Currency substitution can then be measured as the elasticity of substitution between home and foreign goods.

utility function it is not possible to meaningfully evaluate the impact of substitutability on welfare. The approach here is more similar in spirit to that of Guidotti (1993), who assumes a transactions technology that makes it costly to use currency, or Végh (1989a), where the availability of a foreign currency reduces transactions costs. 1/

We apply our model to a seignorage tax in the presence of currency substitution. Much of the public finance literature assumes that costless, lump-sum income taxes are possible so that optimal seignorage is zero. 2/ However, in the absence of such income taxes or in the presence of some other distortion, optimal inflation taxes may be positive. 3/ Then the degree of currency substitution will influence tax collection.

We assume that governments must finance their expenditures by levying a costly income tax or by collecting seignorage. In deciding the optimal level of inflation, governments weigh the tradeoffs between the costs of administering and complying with income taxes and the distortions created by rising prices. If identical governments cooperate, then the optimal rate of money growth is strictly positive. It is decreasing in the substitutability of currencies if and only if first- and second-period consumption are gross substitutes. If governments act independently money growth may be either too high or too low.

The intuition is that there are three spillover effects of an increase in a country's money growth. First, the seignorage tax is borne by consumers in both countries. Second, higher money growth in the home country allows the home income tax to be lowered. This increases demand for both monies and foreign seignorage rises. Third, because spot market trading is costly and the value of home money declines relative to foreign money, the demand for foreign goods rises relative to the demand for home goods. The relative price of the foreign good rises and so does the wealth of foreign consumers. The first of these spillover effects is negative and the other two are positive. If the two currencies are poor substitutes, the

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1/ Another approach is that of Canto and Nickelsburg (1987) who develop a model where money is solely a store of value. Imperfect currency substitution arises in their model from the possibility that the government may impose capital controls.

2/ In this case, Hercowitz and Sadka (1987) point out that the degree of currency substitution is immaterial for optimal policy.

3/ Végh (1989c) and Sibert (1992) assume income taxation is costly, Canzoneri and Rogers (1990) consider the possibility of non-taxable sectors, and Végh (1989a) considers a distortion introduced by positive foreign interest rates and a transactions technology.

In practice, seignorage has been a substantial source of revenue for many European countries. According to Cukierman, Edwards and Tabellini (1989), seignorage as a fraction of total tax revenue was 12.5 percent for Italy, 9.1 percent for Spain, and 16.6 percent for Portugal during the 1970s and early 1980s.

temptation to extract tax revenue from foreigners is high and the first effect dominates. Uncoordinated policy making leads to too high money growth. However, if the currencies are good substitutes, it is difficult to collect seignorage from foreigners and the latter two effects prevail. Uncoordinated money growth is suboptimally low. This result contrasts sharply with the prevailing belief of policy makers and academics that currency substitution is beneficial because it reduces money growth. ^{1/}

Higher substitutability lowers resource costs; hence with coordinated policies it raises welfare. However, increased substitutability may also increase the spillover costs associated with uncoordinated policies. In this case a numerical example indicates that utility need not be a monotonic function of substitutability.

In section II we present the basic model and analyze the properties of the resulting money demand functions. In section III we analyze the equilibrium and in section IV we present a model of seignorage taxation. Section V is the conclusion.

II. The Model

1. The consumers

There are two countries, 1 and 2. Each country is specialized in the exogenous production of a tradable consumption good. The countries are inhabited by overlapping generations of two-period-lived agents. Agents consume both goods and save the two country-specific monies. In their second period of life, consumers must purchase goods with the seller's currency. However, they may trade in the spot market at a constant proportional cost. ^{2/} A more realistic model would allow for interest-bearing assets and money would be held because of its liquidity. However, because we wish to focus on currency substitution in the simplest possible environment, we follow Kareken and Wallace (1981) and Canto and Nickelsburg (1987) in supposing that money is the only financial asset.

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^{1/} For example, the United Kingdom's (HM Treasury 1989) proposal for an evolutionary approach to monetary union claims that competition between currencies is beneficial because it forces a low rate of inflation.

^{2/} It would be easy to add costly spot market trading for young agents. However, this would only act as a lump-sum tax on their initial wealth. It would be more realistic to assume that there is a fixed cost to entering currency markets and a declining marginal cost. Sturzenegger (1992) assumes a fixed cost to using foreign currency. In this case only wealthier consumers decide to do so and he demonstrates that this makes the inflation tax regressive.

It is supposed that consumers are uncertain about their second-period preferences when they are young. This may reflect stochastic tastes or it may be that the desire to consume a particular good depends upon some other exogenous event not captured in the model. It is only when they are old that they learn of their exact preferences. Thus, a consumer must decide how much of each money to hold prior to knowing his demand for each good. ^{1/}

The opportunities of the generation- t , country- i consumer when old are represented by the kinked budget set in Figure 1. Let $c_{j,t+1}^{oi}$ be his consumption of good j ($j=1,2$) at time $t+1$, e_{t+1} be the time- $t+1$ price of money 2 in terms of money 1, $p_{j,t+1}$ be the time- $t+1$ price of good j in terms of money 1, and $h-1 > 0$ be the proportional cost of spot market transactions. Suppose the consumer enters his second period of life with m_{jt}^i units of country- j money. If he does not trade in the spot market for foreign exchange then his consumption is indicated by point B. If he engages in costly spot market trading of money 2 for money 1, he can consume along line segment AB; if he trades money 2 for money 1 he can consume along BC.

The uncertainty faced by consumers is represented formally as follows. In each period $t = \dots, -1, 0, 1, \dots$ a unit interval of agents is born in each country. Agents are indexed by their location in the unit interval. An agent's location, α , is unknown to him when young; however, he believes that it is distributed uniformly on $(0,1)$.

The preferences of a consumer are given by

$$W^i = U(c_{1t}^{yi}, c_{2t}^{yi}; .5) + \beta E_t[U(c_{1,t+1}^{oi}, c_{2,t+1}^{oi}; \alpha)], \quad (1)$$

where c_{jt}^{yi} and $c_{j,t+1}^{oi}$ are consumption of good j , $j=1,2$, when young and when old, respectively, and

$$U(c_1, c_2; \alpha) = \begin{cases} (c_1^\alpha c_2^{1-\alpha})^{1-\mu} / (1-\mu) & \text{if } 1 \neq \mu > 0 \\ \alpha \ln c_1 + (1-\alpha) \ln c_2 & \text{if } \mu = 1. \end{cases} \quad (2)$$

When a country- i resident is young he is endowed with x units of his country's consumption good. He pays a real tax $\tau^1 \in (0,x)$, and allocates his after-tax income between consumption and savings of home and foreign money. Thus, his budget constraint when young is

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^{1/} The model is related to Goldman (1974). There agents are willing to hold non-interest-bearing money in addition to bonds because they are unsure of their discount rate and portfolio adjustment is costly.

Figure 1

$$p_{1t}c_{1t}^{yi} + p_{2t}c_{2t}^{yi} + m_{1t}^i + e_tm_{2t}^i = p_{it}(x - \tau_t^i). \quad (3)$$

When the agent is old, he uses his savings to purchase the consumption goods. His budget constraint is

$$\begin{aligned} hp_{1,t+1}c_{1,t+1}^{oi} + p_{2,t+1}c_{2,t+1}^{oi} &= hm_{1t}^i + e_{t+1}m_{2t}^i \\ \text{if } p_{1,t+1}c_{1,t+1} &\geq m_{1t}^i \\ p_{1,t+1}c_{1,t+1}^{oi} + hp_{2,t+1}c_{2,t+1} &= m_{1t}^i + he_{t+1}m_{2t}^i \\ \text{otherwise.} \end{aligned} \quad (4)$$

At time $t+1$, each generation- t agent learns his value of α and maximizes $U(c_{1,t+1}^{oi}, c_{2,t+1}^{oi}; \alpha)$ subject to (4). The solution has

$$c_1^{oi'} = \begin{cases} \bar{c}_1^{oi'} = \frac{\alpha m_1^i}{\bar{\alpha}^{i'} p_1'} & \text{if } \alpha > \bar{\alpha}^{i'} \\ \underline{c}_1^{oi'} = \frac{\alpha m_1^i}{\underline{\alpha}^{i'} p_1'} & \text{if } \alpha < \underline{\alpha}^{i'} \\ m_1^i / p_1' & \text{otherwise} \end{cases}; \quad c_2^{oi'} = \begin{cases} \bar{c}_2^{oi'} = \frac{(1 - \alpha) e' m_2^i}{(1 - \bar{\alpha}^{i'}) p_2'} & \text{if } \alpha > \bar{\alpha}^{i'} \\ \underline{c}_2^{oi'} = \frac{(1 - \alpha) e' m_2^i}{(1 - \underline{\alpha}^{i'}) p_2'} & \text{if } \alpha < \underline{\alpha}^{i'} \\ e' m_2^i / p_2' & \text{otherwise,} \end{cases} \quad (5)$$

where variables without primes are evaluated at time t , variables with a prime are evaluated at time $t+1$, and

$$\bar{\alpha}^{i'} = hm_1^i / (hm_1^i + e' m_2^i); \quad \underline{\alpha}^{i'} = m_1^i / (m_1^i + he' m_2^i). \quad (6)$$

Consumption when young satisfies

$$c_j^{yi} = [p_i(x - \tau^i) - m_1^i - em_2^i] / (2p_j). \quad (7)$$

Thus, the agent chooses money holdings to maximize

$$W^i = U(c_1^{yi}, c_2^{yi}; .5) + \beta \left[\int_0^{\alpha^{i'}} U(\underline{c}_1^{oi'}, \underline{c}_2^{oi'}; \alpha) d\alpha + \int_{\alpha^{i'}}^{\bar{\alpha}^{i'}} U\left(\frac{m_1^i}{p_1'}, \frac{e'm_2^i}{p_2'}; \alpha\right) d\alpha + \int_{\bar{\alpha}^{i'}}^1 U(\bar{c}_1^{oi'}, \bar{c}_2^{oi'}; \alpha) d\alpha \right] \quad (8)$$

subject to (5)-(7).

Employing Leibnitz's rule and noting that $U(\underline{c}_1^{oi}, \underline{c}_2^{oi}; \alpha) \rightarrow U(m_1^i/p_1', e'm_2^i/p_2'; \alpha)$ when $\alpha \nearrow \alpha^{i'}$ and $U(\bar{c}_1^{oi}, \bar{c}_2^{oi}; \alpha) \rightarrow U(m_1^i/p_1', e'm_2^i/p_2'; \alpha)$ when $\alpha \searrow \bar{\alpha}^{i'}$ gives the Kuhn-Tucker conditions

$$\begin{aligned} \frac{1}{\beta p_2} U_2(c_1^{yi}, c_2^{yi}; .5) &\geq \int_0^{\alpha^{i'}} \frac{1}{h p_2'} U_2(\underline{c}_1^{oi'}, \underline{c}_2^{oi'}; \alpha) d\alpha \\ &+ \int_{\alpha^{i'}}^{\bar{\alpha}^{i'}} \frac{1}{p_1'} U_1\left(\frac{m_1^i}{p_1'}, \frac{e'm_2^i}{p_2'}; \alpha\right) d\alpha + \int_{\bar{\alpha}^{i'}}^1 \frac{h}{p_2'} U_2(\bar{c}_1^{oi'}, \bar{c}_2^{oi'}; \alpha) d\alpha \end{aligned} \quad (9)$$

with equality if $m_1^i > 0$

$$\begin{aligned} \frac{e}{\beta p_2} U_2(c_1^{yi}, c_2^{yi}; .5) &\geq \int_0^{\alpha^{i'}} \frac{e'}{p_2'} U_2(\underline{c}_1^{oi'}, \underline{c}_2^{oi'}; \alpha) d\alpha \\ &+ \int_{\alpha^{i'}}^{\bar{\alpha}^{i'}} \frac{e'}{p_2'} U_2\left(\frac{m_1^i}{p_1'}, \frac{e'm_2^i}{p_2'}; \alpha\right) d\alpha + \int_{\bar{\alpha}^{i'}}^1 \frac{e'}{p_2'} U_2(\bar{c}_1^{oi'}, \bar{c}_2^{oi'}; \alpha) d\alpha \end{aligned} \quad (10)$$

with equality if $m_2^i > 0$.

2. Money demand

In this section we consider the properties of the money demand functions given by (9) and (10). We first show that consumers will hold both currencies if and only if one currency is not expected to depreciate by too much against the other.

Proposition 1. Suppose that prices and exchange rates are strictly positive and finite and $h > 1$. Then there exists a unique solution to the consumer's problem. The solution has $m_1^i > 0$ if and only if $h > e'/e$ and $m_2^i > 0$ if and only if $h > e/e'$.

Proof. See the Appendix.

The intuition for this result is as follows. If $h < e'/e$, then the return to money 2 (in terms of money 1) exceeds the cost of exchanging money 2 for money 1 in the spot market. Thus, consumers will choose to hold only money 2. Likewise, if $h < e/e'$, then the return to money 1 (in terms of money 2) exceeds the cost of exchanging money 1 for money 2 in the spot market. If $h = e'/e$, then the return to money 2 equals the cost of exchanging money. However, at any interior solution there is some possibility that once α is revealed the consumer will want to purchase more money 2. This would result in less second-period consumption than if only money 2 were held. Likewise, if $h = e/e'$, only money 1 will be held. If h is infinite, both monies will always be held; if $h = 1$, then an interior solution requires non-asset dominance; the exchange rate must be constant.

Proposition 1 does not limit the model to a small band of exchange rate changes for two reasons. First, the model has no aggregate uncertainty. With such uncertainty, relatively small expected and large unexpected changes may be possible. Second, while transactions costs are very small in the interbank market, they may be much larger in the retail market. According to de Larosiere (1991), the European Commission estimates that transactions costs associated with multiple currencies are 0.5 percent of the European Community's GDP.

We now consider the response of money demand to a change in the second-period price of good 2 in terms of good 1. Let $\delta^i = m_1^i / (m_1^i + em_2^i)$ be the fraction of the country- i portfolio allocated to money 1.

Proposition 2. Suppose that $h > \max (e'/e, e/e')$. Then,

$$\frac{\partial \delta^i}{\partial (p_2'/p_1')} \begin{cases} > 0 & \text{if } \mu > 1 \\ = 0 & \text{if } \mu = 1 \\ < 0 & \text{if } \mu < 1 \end{cases} \quad (11)$$

Proof. See the Appendix.

If the relative price of good two falls, this increases the return, in terms of consumption, of money 1. In this case relative demand for money 1 will increase (decrease) if the substitution effect dominates (is dominated by) the income effect. This will be the case if first- and second-period consumption are gross substitutes (compliments).

We would like to be able to say how the money demand functions depend on the appreciation of currency 2, e'/e , and the (gross) transactions cost, h . Unfortunately this is complicated by the presence of income as well as substitution effects and we are able to obtain unambiguous results only for the case of log-linear preferences.

Suppose that $\mu = 1$. If $h > \max \{e'/e, e/e'\}$ then the solution is interior and (9) and (10) hold with equality. In this case, multiplying both sides of (9) by m_1^1 and both sides of (10) by m_2^1 and summing the results yields $m_1^1 + em_2^1 = \sigma(x - r^1)$, where $\sigma = \beta/(1 + \beta) \in (0, .5)$. Thus, total savings is a constant fraction of wealth. It is easy to see that δ^1 is a function solely of e'/e and h . 1/ We show that an increase in the appreciation of the foreign currency has the expected result of lowering the fraction of portfolios allocated to currency 1. An increase in the cost of exchanging money increases the precautionary relative demand for the weaker currency.

Proposition 3. Suppose that $\mu = 1$ and $h > \max \{e'/e, e/e'\}$. Then,

$$\begin{aligned} \partial \delta^1 / \partial (e'/e) &< 0 \\ \partial \delta^1 / \partial h &\begin{cases} > 0 & \text{if } e'/e > 1 \\ = 0 & \text{if } e'/e = 1. \\ < 0 & \text{if } e'/e < 1 \end{cases} \end{aligned} \quad (12)$$

Proof. See the Appendix.

3. The government

Each period government i produces an amount g^i of the public good. This good is produced from the country's private consumption good via a linear production technology. Each unit of the private good can be costlessly transformed into a unit of the public good. The government finances its purchases of inputs by levying income taxes and by collecting seignorage.

It is supposed that income taxation is costly to administer and expensive to comply with. If residents of country i forego r^i units of resources to pay their taxes, the government ends up with only cr^i units of revenue with which to purchase inputs for the public good, where $c < 1$. Thus, the governments' budget constraints are

$$\begin{aligned} g^1 &= (M_1' - M_1) / p_1' + cr^1 \\ g^2 &= e'(M_2' - M_2) / p_2' + cr^2, \end{aligned} \quad (13)$$

where M_j is the stock of money j , $j=1,2$.

1/ To see this, multiply (9) by e . Subtract the left and right sides of (9) from the left and right sides of (10). With log-linear preferences, all of the prices cancel out.

III. Equilibrium

In this section we define an equilibrium and provide conditions for existence.

Money market clearing requires

$$m_j^1 + m_j^2 = M_j, \quad j=1, 2. \quad (14)$$

Market clearing for good 1 requires that spending on good 1 by the young, the old, and the government, plus real resources lost in costly spot market trading and costly tax collection equal the amount of good 1 available:

$$\sum_{i=1}^2 \left(c_1^{y i'} + \int_0^{\bar{\alpha}^{i'}} c_1^{o i'} d\alpha + \int_{\bar{\alpha}^{i'}}^{\bar{\alpha}^{i'}} m_1^i / p_1' d\alpha + \int_{\bar{\alpha}^{i'}}^1 \bar{c}_1^{o i'} d\alpha \right. \\ \left. + (h-1) \int_{\bar{\alpha}^{i'}}^1 (\bar{c}_1^{o i'} - m_1^i / p_1') d\alpha \right) + g + (1-c) \tau_1' = x. \quad (15)$$

The first step in analyzing the equilibrium is to define it in terms of real variables. Let $\gamma_j^i = m_j^i / M_j$ be the fraction of money j held by residents of country i , $z_j = M_j / M_j$ be the reciprocal of country- j money growth, $q = eM_2 / M_1$ be the ratio of money 2 real balances to money 1 real balances, and $r_j = p_j / M_1$ be the reciprocal of the real value of country-1 money in terms of good j .

It is well known that monetary rational expectations models can have a plethora of equilibria; we focus on the one we find most believable. The current state of the world is summarized by the current policy variables; hence it is natural to consider equilibria which are stationary in that real variables depend only on these policies. We will be restricting attention to constant government policies; hence we will only consider constant real variables. In this case, rewriting equations (5) - (7), (9), (10) and (13)-(15) yields

$$c_1^{oi} = \begin{cases} \bar{c}_1^{oi} = \frac{\alpha \gamma_1^i z_1}{\bar{\alpha}^i r_1} & \text{if } \alpha > \bar{\alpha}^i \\ \underline{c}_1^{oi} = \frac{\alpha \gamma_1^i z_1}{\underline{\alpha}^i r_1} & \text{if } \alpha < \underline{\alpha}^i \\ \gamma_1^i z_1 / r_1 & \text{otherwise} \end{cases}; \quad c_2^{oi} = \begin{cases} \bar{c}_2^{oi} = \frac{(1 - \alpha) \gamma_2^i q z_2}{(1 - \bar{\alpha}^i) r_2} & \text{if } \alpha > \bar{\alpha}^i \\ \underline{c}_2^{oi} = \frac{(1 - \alpha) \gamma_2^i q z_2}{(1 - \underline{\alpha}^i) r_2} & \text{if } \alpha < \underline{\alpha}^i \\ \gamma_2^i q z_2 / r_2 & \text{otherwise} \end{cases} \quad (16)$$

$$\bar{\alpha}^i = h \gamma_1^i z_1 / (h \gamma_1^i z_1 + \gamma_2^i q z_2); \quad \underline{\alpha}^i = \gamma_1^i z_1 / (\gamma_1^i z_1 + h \gamma_2^i q z_2) \quad (17)$$

$$c_j^i = [r_i (x - \tau^i) - \gamma_1^i - \gamma_2^i q] / (2 r_j) \quad (18)$$

$$U_2^{yi} / (\beta z_1) = (1/h) \int_0^{\underline{\alpha}^i} U_2^i d\alpha + (r_2/r_1) \int_{\underline{\alpha}^i}^{\bar{\alpha}^i} U_1 d\alpha + h \int_{\bar{\alpha}^i}^1 \bar{U}_2 d\alpha \quad (19)$$

$$U_2^{yi} / (\beta z_2) = \int_0^{\underline{\alpha}^i} U_2^i d\alpha + \int_{\underline{\alpha}^i}^{\bar{\alpha}^i} U_2^i d\alpha + \int_{\bar{\alpha}^i}^1 \bar{U}_2 d\alpha \quad (20)$$

$$g^1 = (1 - z_1) / r_1 + c \tau^1; \quad g^2 = q(1 - z_2) / r_2 + c \tau^2 \quad (21)$$

$$\gamma_j^1 + \gamma_j^2 = 1, \quad j=1,2 \quad (22)$$

$$\sum_{i=1}^2 \left(c_1^{yi} + \int_0^{\underline{\alpha}^i} \underline{c}_1^{oi} d\alpha + \int_{\underline{\alpha}^i}^{\bar{\alpha}^i} \gamma_1^i z_1 / r_1 d\alpha + h \int_{\bar{\alpha}^i}^1 \bar{c}_1^{oi} d\alpha \right. \\ \left. - (h-1) \int_{\bar{\alpha}^i}^1 \gamma_1^i z_1 / r_1 d\alpha \right) + g + (1-c) \tau_1 = x, \quad (23)$$

where $U^{yi} = U(c_1^{yi}, c_2^{yi}; .5)$, $\underline{U}^i = U(\underline{c}_1^{oi}, \underline{c}_2^{oi}; \alpha)$, $\bar{U}^i = U(.5 \gamma_1^i z_1 / r, .5 \gamma_2^i q z_2 / r; \alpha)$, and $\bar{U}^i = U(\bar{c}_1^{oi}, \bar{c}_2^{oi}; \alpha)$.

Def. Given $(z^i, r^i)_{i=1,2}$, an equilibrium is a vector $(((\gamma_j^i, c_j^{yi}, \underline{c}_j^{oi}, \bar{c}_j^{oi})_{j=1,2}, \underline{\alpha}^i, \bar{\alpha}^i, r_i, g^i)_{i=1,2}, q)$ of positive, finite numbers such that $r_i > 0$, $i=1,2$, $q > 0$, and equations (16) - (23) hold.

It is now shown that a stationary equilibrium exists if transactions costs are not too small and monetary policies are not too different.

Proposition 4. A stationary equilibrium exists if and only if $h > \max \{z_1/z_2, z_2/z_1\}$.

Proof. See the Appendix.

If the monetary policies of the two countries are sufficiently different relative to the size of the transactions cost then a stationary equilibrium will not exist. Instead, the value of the money stock of the country with the more conservative monetary policy will proceed to a negligible part of the world money stock. ^{1/}

Proposition 5. If $h \leq \max \{z_1/z_2, z_2/z_1\}$ and there exists a non-stationary equilibrium where both currencies are valued then

$$z_1 > z_2 \Rightarrow q \rightarrow \infty; z_2 > z_1 \Rightarrow q \rightarrow 0. \quad (24)$$

Proof. By Proposition 1, both monies are valued only if $h > \max \{e'/e, e/e'\}$. The result then follows from $e'/e = (q'/q)(z_2/z_1)$.

If transactions costs are zero (h is equal to one), then the model reduces to the Kareken and Wallace (1981) exchange-rate-indeterminacy model. Any constant exchange rate is an equilibrium. A stationary equilibrium requires $z_1 = z_2$, otherwise only one money has value.

In the special case where the two countries pursue identical government policies it is easy to verify that $\gamma_j^i = 1/2$, $q = 1$, and

$$c_1^{oi} = \begin{cases} \bar{c}_1^o = \frac{\alpha z(h+1)}{2hr} & \text{if } \alpha > \bar{\alpha} \\ \underline{c}_1^o = \frac{\alpha z(h+1)}{2r} & \text{if } \alpha < \underline{\alpha} \\ z/(2r) & \text{otherwise} \end{cases}; c_2^{oi} = \begin{cases} \bar{c}_2^o = \frac{(1-\alpha)z(h+1)}{2r} & \text{if } \alpha > \bar{\alpha} \\ \underline{c}_2^o = \frac{(1-\alpha)z(h+1)}{2hr} & \text{if } \alpha < \underline{\alpha} \\ z/(2r) & \text{otherwise} \end{cases} \quad (25)$$

$$\bar{\alpha} = h/(h+1); \underline{\alpha} = 1/(h+1) \quad (26)$$

$$c_j^{yi} = c^y = [r(x - \tau) - 1]/(2r). \quad (27)$$

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^{1/} This is similar to results in Woodford (1991) and Weil (1991).

To find r , note that by a transformation of variables

$$\beta \int_0^{\infty} U_2^i d\alpha = (\beta/h) \int_{\alpha}^1 \bar{U}_2^i d\alpha = (.5z/r)^{-\mu} \Psi(h) h / (h+1), \quad (28)$$

$$\text{where } \Psi(h) = \beta \int_0^{\infty} \left[(1+h) \alpha^{\alpha} \left(\frac{1-\alpha}{h} \right)^{1-\alpha} \right]^{1-\mu} d\alpha.$$

Then by (20),

$$r(x - \tau) - 1 = [2(\Psi + A)]^{-\frac{1}{\mu} z^{-\frac{1-\mu}{\mu}}}, \quad (29)$$

where $A(h) \equiv .5\beta(h-1)/(h+1)$.

IV. Government Finance

In this section we consider a public finance application. We suppose that the two countries are identical and that each period each of the governments is required to produce an amount g of the public good. 1/ Production can be financed through income taxation or by collecting seignorage. The goal of each government is to choose a constant money growth rate and tax to maximize steady-state utility. 2/ By (8), steady state-utility can be written as

$$W^i = U^{yi} + \beta \left(\int_0^{\infty} U^i d\alpha + \int_{\alpha^i}^1 U^i d\alpha + \int_{\alpha^i}^1 \bar{U}^i d\alpha \right). \quad (30)$$

Differentiating (30), employing the first-order conditions $h \bar{V}_1^i / V_2^i = \bar{V}_1^i / (h \bar{V}_2^i) = V_1^i / V_2^i = r_1 / r_2$, (16), (18), (21), and (28) yields that at a symmetric outcome

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1/ This case is somewhat different from the previous section in that g is exogenous, rather than the taxes and money growth rates. However, existence is simplified because we consider only symmetric outcomes.

2/ Confining attention to constant strategies is not unreasonable because the stationarity of the model implies the set of state variables summarizing the economic environment is null; hence Markov strategies would be constant. However, assuming that the government maximizes steady-state utility is ad hoc. It would be preferable, but vastly more complicated, to assume policy makers care about a weighted sum of the utilities of different generations.

$$\frac{2r}{z} dW^i = \frac{U_2^y}{z} [r(x - \tau) - 1] (2\hat{p}_1^{yi} + \hat{p}_1 - \hat{p}_2) + \beta \left[(1/h + 1) \int_{\underline{\alpha}}^1 \bar{U}_2 d\alpha + \int_{\underline{\alpha}}^{\bar{\alpha}} U_2 d\alpha \right] (\hat{q}_1^i + \hat{q}_2^i + \hat{z}_1 + \hat{z}_2 + \hat{q} - \hat{p}_1 - \hat{p}_2), \quad (31)$$

where a "hat" over a variable denotes a proportional change. By (20) and (28),

$$U_2^y / (\beta z) = (1/h + 1) \int_{\underline{\alpha}}^1 \bar{U}_2 d\alpha + \int_{\underline{\alpha}}^{\bar{\alpha}} U_2 d\alpha; \quad (32)$$

hence, by (18), (21) and (31), at a symmetric outcome

$$\begin{aligned} \frac{2r dW^1}{U_2^y} &= r(x - \tau) (\hat{p}_1 - \hat{p}_2) - \frac{2(1 - z)}{c} \hat{p}_1 + \hat{z}_2 + \frac{c - 2z}{c} \hat{z}_1 \\ \frac{2r dW^2}{U_2^y} &= r(x - \tau) (\hat{p}_2 - \hat{p}_1) - \frac{2(1 - z)}{c} (\hat{p}_2 - \hat{q}) + \hat{z}_1 + \frac{c - 2z}{c} \hat{z}_2. \end{aligned} \quad (33)$$

1. Optimal money growth

As a bench mark we consider the case where the two countries cooperate. Because the countries are posited to be identical it seems reasonable to consider the case where they are treated equally and the outcome is symmetric. In this case (33) implies that the coordinated outcome has

$$c - z = (1 - z) \left(\frac{\partial r_1}{\partial z_1} \frac{z}{r} + \frac{\partial r_2}{\partial z_1} \frac{z}{r} - \frac{\partial q}{\partial z_1} \frac{z}{q} \right); \quad i=1, 2. \quad (34)$$

Straightforward, but tedious, differentiation of (16), (18) and (19) - (22) yields

$$\hat{p}_1 + \hat{p}_2 - \hat{q} = - \frac{c(1 - \mu) [r(x - \tau) - 1] - \mu z}{\mu [cr(x - \tau) - 1 + z]} (\hat{z}_1 + \hat{z}_2). \quad (35)$$

Substituting (35) into (34) implies that an interior solution to the social welfare problem is

$$z = 1 - \mu(1 - c)r(x - \tau) / [r(x - \tau) - 1] < 1, \quad (36)$$

if $c < 1$.

Thus, if income taxes are costly, it is optimal to have positive money growth. It is possible that an interior solution does not exist and it is optimal to have zero income taxes. This would be the case if taxes were so costly to collect that governments would prefer to finance all spending by inflation taxation. By (29), $r(x - \tau)$ does not depend upon c . Therefore, by (21) and (36), an interior solution exists if c is not too small and $g < rx$.

For the special case of $\mu = 1$, (29) implies that

$$r(x - \tau) = 1/\sigma \text{ where } \sigma \equiv \beta/(1 + \beta) \quad (37)$$

Thus, by (36) and (37),

$$z = 1 - (1 - c)/(1 - \sigma). \quad (38)$$

In this case $c > \sigma$ and $g < rx$ ensure that an interior solution exists.

We now show how optimal monetary policy varies with currency substitutability.

Proposition 5. Optimal money growth is strictly increasing (decreasing) in currency substitutability if $\mu > (<) 1$.

Proof. See the Appendix.

The intuition is that if $\mu < 1$, then first- and second-period consumption are gross substitutes; an increase in the return to savings increases savings. In this case an increase in a country's money growth decreases the demand for savings and lowers the seignorage revenue associated with a given amount of money growth. This effect is strengthened if the two currencies become better substitutes. On the other hand, if $\mu > 1$, then first- and second period consumption are gross complements and an increase in the return to savings decreases savings and the seignorage revenue associated with a given amount of money growth. This effect is enhanced if the two currencies become better substitutes. If $\mu = 1$, the income and substitution effects cancel out; savings does not depend on the return and substitutability does not affect the optimal money growth.

This result is in contrast to Végh (1989a,b). Végh (1989a) supposes that the government can impose a consumption tax or inflate. Foreign currency in his model reduces transactions costs, but because the foreign interest rate is positive, consumers do not hold enough. This distortion increases the effective cost of consumption. The inflation tax ought to be positive because this lowers the necessity to tax consumption and decreases consumption costs. Végh (1989b) demonstrates that the greater is substitutability, the greater is

the distortion caused by the positive foreign interest rate and the higher is optimal home inflation. Our results are more in the spirit of Imrohorglu (1991), who finds the Laffer curve becomes more peaked as substitution rises.

An increase in currency substitutability lowers the resource costs associated with transacting. With coordinated policy making this increases welfare. ^{1/}

Proposition 6. With coordinated policy making, a decrease in transactions costs increases welfare.

Proof. See the Appendix.

2. The uncoordinated case

We now consider the case where countries do not cooperate. As this case is less tractable, we restrict our attention to $\mu = 1$. Suppose that each country's central bank takes the other country's money growth as given and chooses a constant money growth rate to maximize steady-state utility. Then by (33)

$$\begin{aligned} 0 = r(x - \tau) \left(\frac{\partial r_1}{\partial z_1} \frac{z}{r} - \frac{\partial r_2}{\partial z_1} \frac{z}{r} \right) - \frac{2(1-z)}{c} \frac{\partial r_1}{\partial z_1} \frac{z}{r} + \frac{c-2z}{c} = \\ r(x - \tau) \left(\frac{\partial r_2}{\partial z_2} \frac{z}{r} - \frac{\partial r_1}{\partial z_2} \frac{z}{r} \right) - \frac{2(1-z)}{c} \left(\frac{\partial r_2}{\partial z_2} \frac{z}{r} - \frac{\partial q}{\partial z_2} \frac{z}{q} \right) + \frac{c-2z}{c}. \end{aligned} \quad (39)$$

By (37), (39) and symmetry,

$$\frac{c}{\sigma} \left(\frac{\partial r_1}{\partial z_1} \frac{z}{r} - \frac{\partial r_2}{\partial z_1} \frac{z}{r} \right) - 2(1-z) \frac{\partial r_1}{\partial z_1} \frac{z}{r} + c - 2z = 0. \quad (40)$$

Brute-force differentiation of (16) and (18)-(23) yields

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^{1/} Tanzi and Blejer (1982) note that increased substitutability may be harmful in that it lessens the central bank's ability to stabilize.

$$f_1 + f_2 = \frac{\sigma z}{c - \sigma + \sigma z} (\hat{z}_1 + \hat{z}_2) + A(\hat{z}_1 - \hat{z}_2) \quad (41)$$

$$2f_1 = \frac{c[\sigma z(1 - B)(A - 1) + (1 - \sigma)A](\hat{z}_1 - \hat{z}_2) + 2\sigma z\hat{z}_1}{c - \sigma + \sigma z}, \quad (42)$$

where

$$A \equiv \frac{-8h}{(h - 1)(h^2 + 4h - 1)}; \quad B \equiv \frac{h(h - 1)}{(h + 1)^2}. \quad (43)$$

Substituting (41) and (42) into (40) yields

$$1 - \frac{z}{c - \sigma + \sigma z} = A - z(1 - B)(A - 1). \quad (44)$$

The left-hand side (LHS) of (44) is strictly decreasing in z , going to one as z goes to zero and to $-(1-\sigma)/\sigma$ as z goes to infinity. By (43), $A \leq 0$ and $B \leq 1$ with strict inequality if h is finite; hence the right-hand side (RHS) of (44) is strictly increasing in z , becoming negative as z goes to zero and going to infinity as z goes to infinity. This is illustrated in Figure 2. At $z = 1$, $\text{LHS} = (c-1)/c < 0 < \text{RHS}$; hence it is clear from the Figure that there exists a unique $z^n \in (0,1)$ such that (44) is satisfied. In Figure 2, Nash money growth is lower than is optimal but Proposition 6 demonstrates that this need not be the case.

Let $D(h) \equiv 8h/[(3h + 1)/(h + 1)] \in (0,1)$.

Proposition 7. If transactions costs are finite then uncoordinated policy making leads to suboptimally low (high) money growth if $(c - \sigma)/(1 - \sigma) < (>) D(h)$. If transactions costs are infinite, uncoordinated policy making leads to optimal money growth.

Proof. Suppose $h < \infty$ and let z^0 be optimal money growth, as given by (38). By Figure 2 it is clear that $z^n > z^0$ if and only if $\text{LHS}(z^0) = 0 > \text{RHS}(z^0)$. By (43) and (44) this is true if and only if $(c - \sigma)/(1 - \sigma) < D(h)$. Suppose $h \rightarrow \infty$, then $A \rightarrow 0$ and $B \rightarrow 1$; hence $\text{RHS} = 0$.

The intuition is as follows. Suppose that country 2 increases its money growth. There are three spillover effects. First, old consumers in country 1 as well as country 2 are made poorer; part of burden of a seignorage tax can be shifted to foreigners. Second, the country-2 income tax can be lowered;

Figure 2

hence, the wealth of the country 2 young rises and so does their demand for currency 1. Thus, country 1's seignorage revenue rises as well. Third, the decline in the purchasing power of currency 2 coupled with the existence of transactions costs means that the demand for good 2 falls relative to good 1. Hence the relative price of good 1 rises and the country 1 consumers become wealthier.

The first effect is negative and the other two are positive. Which effect dominates depends upon how substitutable the currencies are. If transactions costs are high, then neither home nor foreign consumers will want to react to an increase in country 2 money growth by shifting most of their portfolios to money 1. This means that the first effect dominates. This temptation to inflate too much in an attempt to tax foreigners is similar to the result in Canzoneri (1989). If transactions cost are low, consumers can readily shift their portfolios and the latter two affects dominate. The result that money growth is too low and taxes are too high is similar to the result of Sibert (1992) for uncoordinated fiscal policy in a monetary union.

We would like to find the impact of currency substitutability on uncoordinated policy making. However, while a sufficiently low degree of substitutability guarantees that the first of our three spillover effects dominates, it is not the case the decreasing substitutability always increases uncoordinated money growth. In fact, as Proposition 7 shows, the optimal policy can be attained when transactions costs become infinite. This ambiguity occurs because each of our three spillover effects becomes more important as substitutability falls. A partial result is available.

Proposition 8. Suppose that money growth is suboptimally low. As transactions costs rise, money growth with uncoordinated policy making moves closer to its optimal level.

Proof. See the Appendix.

Figure 3 depicts z^n as a function of h for the special case of $\sigma = .47$ and $c = .8$. In this case it is seen that z^n is decreasing in h for low transactions costs and increasing otherwise.

With uncoordinated policies, increased currency substitution can worsen the monetary policy spillover. It is possible that this effect may dominate the effect of lower transactions costs. Thus, welfare need not be monotonic in h . This is seen in Figure 4 for the special case of $\sigma = .47$, $c = .8$, and $g/x = .3$.

V. Conclusion

This paper develops a model of currency substitution. It is shown that the money demand functions have reasonable properties. Demand for both currencies is positive when the cost of transacting in the spot market exceeds the rate of depreciation for the weaker currency. An increase in the future price of country two's good relative to country one's good decreases the share of portfolios allocated to the currency of country one if and only if first- and second-period consumption are gross substitutes. If utility is log-linear, the share of portfolios allocated to the weaker (stronger) currency depends positively (negatively) upon the transactions cost. The share of portfolios allocated to either currency depends positively upon its rate of appreciation. A stationary equilibrium is shown to exist if the cost of spot market transactions is not too small and monetary policies are not too different.

The model is applied to seignorage taxation. If countries coordinate their monetary policy, it is shown that with costly income taxation, optimal money growth is strictly positive. It is decreasing in currency substitutability if and only if first- and second-period consumption are gross substitutes. If countries do not coordinate their monetary policy and preferences are log linear, money growth may be too high or too low depending on the degree of currency substitutability.

Proof of Proposition 1. It is straightforward to demonstrate that W^i is strictly concave. Let $b^i = m_1^i + em_2^i$ and suppose that $\delta^i = m_1^i / (m_1^i + em_2^i) \in [0,1]$. By (5) - (7), the right-hand side of (10) is strictly decreasing in b^i ; going to infinity as $b^i \rightarrow 0$ and to zero as $b^i \rightarrow \infty$. The left-hand side of (10) is strictly increasing; going to infinity as $b^i \rightarrow p_i(x - \tau^i)$ and to a strictly positive constant as $b^i \rightarrow 0$. Then by the implicit function theorem, there exist $b^i = b^i(\delta^i)$ such that (10) holds with equality.

Multiplying both sides of (10) by $1/e$ and subtracting both sides of (9) from the result yields

$$L = \int_0^{\alpha^{i'}} \frac{sh-1}{hp_2^{i'}} U_2(\bar{c}_1^{oi'}, \bar{c}_2^{oi'}; \alpha) d\alpha + \int_{\alpha^{i'}}^{\bar{\alpha}^{i'}} \left[\frac{s}{p_2'} U_2\left(\frac{m_1^i}{p_1'}, \frac{e'm_2^i}{p_2'}; \alpha\right) - \frac{1}{p_1'} U_1\left(\frac{m_1^i}{p_1'}, \frac{e'm_2^i}{p_2'}; \alpha\right) \right] d\alpha + \int_{\bar{\alpha}^{i'}}^1 \frac{s-h}{p_2'} U_2(\bar{c}_1^{oi'}, \bar{c}_2^{oi'}; \alpha) d\alpha = 0. \quad (A1)$$

where $s = e'/e$.

Figure 3
($\sigma = .47$, $c = .8$)

Figure 4
($\sigma = .47, c = .8, g/x = .3$)

As $\delta^i \rightarrow 0$, W^i remains strictly positive and α^i , and $a_i \rightarrow 0$. Thus $L \rightarrow \int_0^1 (s - h)/p_2' U_2^i d\alpha \rightarrow 0$ as $h \rightarrow s$. As $\delta^i \rightarrow 1$, W^i remains strictly positive and α^i , and $\alpha^i \rightarrow 1$. Thus, $L \rightarrow \int_0^1 (sh - 1)/(hp_2') U_2^i d\alpha \rightarrow 0$ as $h \rightarrow 1/s$. L is continuous and strictly increasing in δ^i , and $h > 1$, hence there is a unique solution to (9) and (10) with equality if and only if $h > \max(s, 1/s)$.

Suppose $1/s > h > s$. Then a solution to (9) and (10) must have δ^i equal to zero or one. If $\delta^i = 0$, then (10) must hold with equality. $L < 0$ implies the right-hand side of (9) is strictly greater than the left-hand side, which is a contradiction. Hence, the only possibility is $\delta^i = 1$. If this is true, (9) holds with equality and $L < 0$. Thus, (10) is satisfied with inequality. The case of $s > h > 1/s$ is similar.

Proof of Proposition 2. Given the hypothesis, (9) and (10) hold with equality. Multiplying both sides of (9) by e subtracting the right- and left-hand sides of (10) from the right- and left-hand sides of the result, canceling out the price of good 2 and using (2) and (5) to evaluate the marginal utilities yields

$$L = (1 - \mu) \left[\frac{sh - 1}{\delta^i + sh(1 - \delta^i)} \int_0^{\alpha^i} U^i d\alpha + \int_{\alpha^i}^1 \left(\frac{1 - \alpha}{1 - \delta^i} - \frac{\alpha}{\delta^i} \right) U^i d\alpha \right. \\ \left. + \frac{s - h}{h\delta^i + s(1 - \delta^i)} \int_{\alpha^i}^1 \bar{U}^i d\alpha \right] = 0. \quad (A2)$$

Again employing (5) and (6) yields

$$0 = (hs - 1) [hs - (hs - 1)\delta^i]^{-\mu} \int_0^{\alpha^i} R \left[\alpha^s \left(\frac{1 - \alpha}{h} \right)^{1-\alpha} \right]^{1-\mu} d\alpha + \\ \int_{\alpha^i}^{\alpha^{i'}} \left(\frac{1 - \alpha}{1 - \delta^i} - \frac{\alpha}{\delta^i} \right) R \{ \delta^{i^s} [s(1 - \delta^i)]^{1-\alpha} \}^{1-\mu} d\alpha \\ + (s - h) [s - (s - h)\delta^i]^{-\mu} \int_{\alpha^{i'}}^1 R \left[\left(\frac{\alpha}{h} \right)^s (1 - \alpha)^{1-\alpha} \right]^{1-\mu} d\alpha, \quad (A3)$$

where $R = (r_2/r_1)^{\alpha(1-\mu)}$. Differentiating the above expression yields the result.

Proof of Proposition 3. Multiply both sides of equation (9) by e and subtract (10) from the result. Then evaluating utilities using (2) and $\mu = 1$ yields

$$A = \int_0^{\alpha^{i'}} \frac{(1 - \alpha)(e - e'h) d\alpha}{h p_2' \bar{c}_2^{oi'}} + \int_{\alpha^{i'}}^{\bar{\alpha}^{i'}} \left(\frac{\alpha e}{m_1^i} - \frac{1 - \alpha}{m_2^i} \right) d\alpha$$

$$+ \int_{\bar{\alpha}^{i'}}^1 \frac{(1 - \alpha)(he - e') d\alpha}{p_2' \bar{c}_2^{oi'}} = 0. \quad (A4)$$

Substituting (5) into (A4) yields

$$A = \int_0^{\alpha^{i'}} \frac{(1 - hs) d\alpha}{\delta^i + hs(1 - \delta^i)} + \int_{\alpha^{i'}}^{\bar{\alpha}^{i'}} \left(\frac{\alpha}{\delta^i} - \frac{1 - \alpha}{1 - \delta^i} \right) d\alpha$$

$$+ \int_{\bar{\alpha}^{i'}}^1 \frac{(h - s) d\alpha}{h\delta^i + s(1 - \delta^i)} = 0. \quad (A5)$$

Using Leibnitz's rule to differentiate A yields $\partial A / \partial \delta^i < 0$, $\partial A / \partial s < 0$ and if and only if $s(1 - \delta^i) - \delta^i < (=) 0$. This yields $\partial \delta^i / \partial s < 0$. It is easy to verify that $\delta^i < (=) .5$ when $s < (=) 1$.

$$\frac{1}{s} \frac{\partial A}{\partial h} = - \frac{\delta^i}{[\delta^i + hs(1 - \delta^i)]^3} + \frac{s(1 - \delta^i)}{[h\delta^i + s(1 - \delta^i)]^3} > 0 \quad (A6)$$

Proof of Proposition 4. Suppose $0 < r_2/r_1$, $1/r_1$, $q/r_2 < \infty$. If $h \leq \max \{z_1/z_2, z_2/z_1\}$ the arguments in the proof of Proposition 1 ensure there is no stationary equilibrium with positive demand for both currencies. Otherwise, by the same argument as in the proof of Proposition 1, equations (20) and (21) can be solved to find

$$\gamma_1^i/r_1 = f^i(r_2/r_1) \in \begin{cases} (0, x - \tau_1) & \text{if } i=1 \\ (0, (r_2/r_1)(x - \tau_2)) & \text{if } i=2 \end{cases} \quad (A7)$$

$$g_2^i/r_2 = g^i(r_2/r_1) \in \begin{cases} (0, x - \tau_1) & \text{if } i=1 \\ (0, (r_2/r_1)(x - \tau_2)) & \text{if } i=2. \end{cases}$$

where f_i and g_i , $i=1,2$, are continuous functions.

Thus, by (22),

$$1/r_1 = \sum_{i=1,2} f^i(r_2/r_1) \in (0, \infty); \quad q/r_2 = \sum_{i=1,2} g^i(r_2/r_1) \in (0, \infty). \quad (A8)$$

Let D denote aggregate demand for good 1, given by the left-hand side of (23). By the above results, (16) - (18), (22), and (23), it can be seen that D is a continuous function of r_2/r_1 . Suppose that $r_2/r_1 \rightarrow \infty$. We show that $D \rightarrow \infty$.

Case (a). $q\gamma_2^2 \rightarrow r_2(x - \tau^2)$. Then $\bar{\alpha}^2 \rightarrow 0$, $\gamma_1^2 \rightarrow 0$ and

$$D \geq h \int_{\frac{\alpha}{2}}^1 \bar{c}_1^{\alpha^2} d\alpha - (h-1) \int_{\frac{\alpha}{2}}^1 \gamma_1^2 z_1/r_1 d\alpha \rightarrow \infty.$$

Case (b). $q\gamma_2^2 \neq r_2(x - \tau^2)$. Then $D \geq c_1^2 \rightarrow \infty$.

Suppose that $r_2/r_1 \rightarrow 0$. We show that D remains strictly less than x. By (18), $\gamma_1^2/r_1 \rightarrow 0$.

Case (a). $\gamma_1^1/r_1 \rightarrow 0$. Then $1/r_1 \rightarrow 0$, $c\tau_1 \rightarrow g$ and $c_1^i \rightarrow 0$, $i=1,2$. $c_1^1 < (x - \tau_1)/2$ and $c_1^2 \rightarrow 0$; hence D remains strictly less than $(x - \tau_1)/2 + \tau_1 < x$.

Case (b). $\gamma_1^1/r_1 \neq 0$. Then $h > \max\{z_1/z_2, z_2/z_1\}$ ensures that country 1 demand for good 1 is less than if only money 1 were demanded. Country 1 has strictly positive income and the relative price of good 2 is finite; hence a strictly positive amount of good 2 must be demanded by country 1 and the demand for good 1 remains strictly less than if this were not so. Thus, D remains strictly less than $x - \tau_1 - 1/r_1 + z_1/r_1 + g + (1 - c)\tau_1 = x$.

Thus, (23) can be solved for a strictly positive, finite r_2/r_1 . This and (A8) ensures strictly positive, finite values of r_1 , r_2 , and q . Then the existence of the other equilibrium variables is straightforward.

Proof of Proposition 5. By (36), z is strictly increasing in $r(x - \tau)$. By (29), $r(x - \tau)$ is strictly increasing (decreasing) in h if and only if $\Psi' + A' =$

$$\frac{\beta(1-\mu)(h+1)^{-\mu}}{h} \int_0^{\frac{1}{h+1}} [\alpha h - (1-\alpha)] [\alpha^{\alpha} (\frac{1-\alpha}{h})^{1-\alpha}]^{1-\mu} d\alpha < (>) 0. \quad (A9)$$

This is true if $\mu > (<) 1$.

Proof of Proposition 6. Differentiation of (16) - (22) reveals that $(\partial r_1/\partial h)(h/r) + (\partial r_2/\partial h)(h/r) + (\partial q/\partial h)(h/q) = 0$. Hence, by the envelope theorem, social welfare is increasing in h if and only if $\Sigma \partial W^i/\partial h > 0$. This is true because $(\partial \bar{c}_1^i/\partial h)(h/\bar{c}_1^i) + (\partial \bar{c}_1^i/\partial h)(h/\bar{c}_1^i) = (h-1)/(h+1) < 0$.

Proof of Proposition 8. Let z^* be such that $RHS = 0$. LHS does not depend on h and RHS is linear in z ; hence it is sufficient to show that z^* moves to the left and RHS evaluated at $z = 0$ moves up as h increases. $z^* = A/[(1 - B)(A - 1)] = 8h/[(3h + 1)(h + 1)]$. Thus, $\partial z^*/\partial h = (1 - 3h^2)/[h(h + 1)(3h + 10)] < 0$; hence z^* moves left. When $z = 0$, $RHS = -8h/[(h - 1)(h^2 + 4h - 1)]$, which is strictly increasing in $h \in (0, \infty)$. Thus, RHS evaluated at $z = 0$ moves up.

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